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Hidden symmetries associated with the projective group of nonlinear first-order ordinary differential equations

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Abstract. Hidden symmetries, those not found by the classical Lie group method for point symmetries, are reported for nonlinear first-order ordinary differential equations (ODEs) which arise frequently in physical problems. These are for the special class of the eight non-Abelian, two-parameter subgroups of the eight-parameter projective group. The first-order ODEs can be transformed by non-local transformations to new separable first-order ODEs which then can be reduced to quadratures. The first-order ODEs include Riccati equations and equations which in particular cases are of the form of Abel's equation. The procedure demonstrates the feasibility of integrating nonlinear ODEs that do not show any apparent Lie group point symmetry. Applications to the Vlasov characteristic equation and the reaction-diffusion equation are given.

1. Introduction

The application of Lie group symmetries to nonlinear differential and differential-integral equations has produced a large number of exact solutions or simplifications of the equations [1-24]. This approach has been widely used in the solution of differential equations which describe physical and engineering as well as biological and chemical problems. It has been especially helpful for nonlinear differential equations as so many of the methods for linear differential equations fail for nonlinear differential equations. Nonlinear diffusion problems are especially amenable to this approach [3, 7, 9] but applications to other nonlinear problems, to general relativity equations [11], fluid equations [5, 8, 12, 13], shocks [14], kinetic equations for plasmas [15-22], transport equations in semiconductors [23], are known.

Not all exact solutions of nonlinear differential equations are found by what has come to be called the classical Lie group method for point symmetries, however. First, the equations may have contact or generalized symmetries. These are not really what is meant here by hidden symmetries although in the case of the higher-order ordinary differential equations (ODEs), symmetries may be missed that are found if the higher-order ODEs are replaced by a set of first-order ODEs [8, 22]. In this analysis hidden symmetries are those that are not found by the usual methods for determining the Lie point, contact or generalized symmetries of a particular differential equation [24]. The hidden symmetries may be the non-classical symmetries for partial differential equations where a PDE, which is equivalent to the new symmetry, is adjoined to the original PDE such that the resultant set is now invariant under the new symmetry [25-27]. Potential symmetries for PDEs have also yielded some interesting solutions [9]. In this work we restrict our considerations to ODEs, which are mostly nonlinear, and consider one sort of hidden symmetry. This symmetry arises when an ODE is converted to a higher-order ODE which is then invariant under a Lie point group that was not an invariance of the original ODE and is called a type I hidden symmetry.

The symmetry is hidden since application of the usual classical method does not find it. That method applies the extended group generator to the differential equation(s) as an invariance condition and requires that the differential equation(s) hold simultaneously. The invariance condition, which is an identity, leads to a set of linear partial differential equations, called determining equations, for the coordinate functions of the group generator. The coordinate transformation, which simplifies the differential equation such that the order can be reduced, is calculated from the characteristic equations for the coordinate functions. Except for the coordinate functions of the group generator for nonlinear first-order ODEs the determining equations can usually be integrated. Integration of the characteristic equations is less certain but for many common groups can be performed. The determination of the Lie point symmetries of a nonlinear first-order ODE is difficult since the determining equation is a single intricate PDE. Most of the nonlinear first-order ODEs that can be identified as invariant under a group were integrated by a guess or were found to be of the general form of a differential equation invariant under a particular Lie point group [1]. The compilation of a table of general forms of nonlinear differential equations invariant under a hidden symmetry is opposite in approach to the direct construction of the Lie point group for a particular nonlinear ODE by the classical method. Calculating the general form of the differential equation from the Lie group symmetries is the inverse problem. This indirect approach is used here for the calculation of the first-order ODEs invariant under hidden symmetries. It was used by Cohen [1] in the early years of this century for Lie point groups and more recently by Fushchich and Nikitin [24].

The hidden symmetries of first-order ODEs are chosen for two reasons. First, the determination of the Lie point group under which first-order ODEs are invariant is the most intractable problem of determining symmetries of ODEs by the direct classical method so that some procedure to enlarge the class of solutions is welcome. Second, one of us has been looking for more solutions of the motion of a one-dimensional charged particle in a time- and space-dependent electric field. This problem arose in the solution of the one-dimensional, nonlinear Vlasov–Maxwell equations for a collisionless plasma [15–18]. Many engineering and physical problems are described by sets of nonlinear, partial differential equations but these are frequently reduced to nonlinear ODEs of first order. Phase plane trajectories have been analysed to determine the behaviour of the solutions [7], but here we look for exact analytical solutions. The simplest case is considered where the higher-order ODEs are of second order and are invariant under a non-Abelian two-parameter group. These second-order ODEs can be reduced to first-order ODEs and then to quadratures if the differential invariants associated with the normal subgroup are chosen. The transformation to canonical coordinates, those coordinates that make the transformed second-order ODE invariant under translations in the independent variable, can also be done but we do not use that approach here. However, if the differential invariants of the non-normal subgroup are chosen, the reduction usually stops in first order [8]. This occurs because a point group symmetry originally present in the second-order ODE has been lost. These reduced, first-order ODEs are the ODEs with the hidden symmetries. For completeness we note that the reduced, first-order ODE may have a group invariance that the original second-order ODE did not possess. The invariance of the first-order ODE under this new group is a hidden symmetry of the second-order ODE and is not discussed here.

The eight-parameter projective group [8–10] has been chosen first since the maximum number of symmetries under which a second-order ODE is invariant is eight. The non-Abelian two-parameter subgroups are selected from the projective group and are found by looking at the commutators of the group generators, partial differential operators, for each of the eight one-parameter subgroups. Eight cases are found by computing the commutators but they may be read from a commutator table [9]. A general form of the first-order ODE is given for each of the eight cases together with the appropriate variable transformation to the associated second-order ODE. The variable transformation to reduce the first-order ODE to quadrature is also indicated. The general procedure for finding all these expressions is shown for one case. Two plasma equations, Vlasov characteristic and reaction–diffusion, are analysed.

2. General procedure for finding hidden symmetries

The group generator is $U = \xi(z, u)\partial/\partial z + \eta(z, u)\partial/\partial u$ for the general second-order ODE,

$$F(z, u, u_z, u_{zz}) = 0 \tag{1}$$

where ξ and η are the coordinate functions and the notation $u_z = \mathrm{d}u/\mathrm{d}z$, $u_{zz} = \mathrm{d}^2u/\mathrm{d}z^2$ is used.

For the projective group the group generators are

$$\begin{aligned} &U_1 = \partial_z \qquad U_2 = \partial_u \qquad U_3 = z\partial_z \qquad U_4 = u\partial_u \qquad U_5 = z\partial_u \\ &U_6 = u\partial_z \qquad U_7 = zu\partial_z + u^2\partial_u \qquad U_8 = z^2\partial_z + zu\partial_u \end{aligned} \tag{2}$$

where the notation $\partial_z = \partial/\partial z$, etc, is introduced for conciseness [8–10].

The Lie algebra associated with a two-parameter symmetry group is defined by the commutator of the group generators. For the non-Abelian two-parameter subgroups of the projective group the commutators of the group generators are of the form

$$[U_\alpha, U_\beta] = kU_\alpha \tag{3}$$

where k is ± 1. The group associated with the group generator U_α is the normal subgroup.

The general form of the second-order ODEs is next given. For the uncomplicated groups treated here the general form of the ODE in second order is first found for each of the eight separate one-parameter groups by calculating the invariants. Next the overlap of the general forms of the second-order ODEs is found. By overlap we mean that the resultant form of the second-order differential equation is valid for both group invariances. Another way to find the general form is by applying the Bluman–Kumei procedure [9]. In that approach the general form of the second-order ODE for the one group is restricted by imposing conditions due to the second group.

The invariants are determined by integrating the characteristic equations of the extended group generator. The characteristic equations are

$$\frac{\mathrm{d}z}{\xi(z, u)} = \frac{\mathrm{d}u}{\eta(z, u)} = \frac{\mathrm{d}u_z}{\eta_z(z, u, u_z)} = \frac{\mathrm{d}u_{zz}}{\eta_{zz}(z, u, u_z, u_{zz})}. \tag{4}$$

The invariants determined from integrating these equations are the first differential invariant, $Y(z, u, u_z)$, and the invariant or path-curve, $X(z, u)$, as well as the second-order differential invariant. The choice of new coordinates x and y which reduce the second-order ODE to a first-order ODE is not unique. We choose $y = Y(z, u, u_z)$ and

$x = X(z, u)$ but we could choose x as a function of X and y as the product of a function of X and a function of Y, for example. The first-order differential equation is of the form

$$\frac{dy}{dx} = f(x, y) \tag{5}$$

and can be found from the second-order differential equation for both subgroups of the two-parameter subgroup. The variable transformation is then determined between the variables for the two first-order ODEs as described in the next section. The solution for the first-order ODE in the non-normal subgroup variables is given by the variable transformation and the solution of the separable ODE in the normal subgroup variables.

3. Hidden symmetries for U_3 and U_8

The procedure is illustrated for one case. We start with the characteristic equations for U_3 and U_8 with commutator $[U_8, U_3] = -U_8$. These are

$$\frac{dz}{z} = \frac{du}{0} = \frac{du_z}{-u_z} = \frac{du_{zz}}{-2u_{zz}} \tag{6}$$

for U_3 and

$$\frac{dz}{z^2} = \frac{du}{zu} = \frac{du_z}{u - zu_z} = \frac{du_{zz}}{-3zu_{zz}} \tag{7}$$

for U_8. These integrate to give the invariants u, zu_z, z^2u_{zz} for U_3 and u/z, $u - zu_z$, z^3u_{zz} for U_8.

The general form of the differential equations is found as a function of the invariants. For these two group generators the general forms of the second-order ODEs are:

$$F(u, zu_z, z^2u_{zz}) = 0$$

for U_3 and

$$F(u/z, u - zu_z, z^3u_{zz}) = 0 \tag{8}$$

for U_8. The overlap of these two ODEs gives

$$F(u - zu_z, z^2uu_{zz}) = 0 \tag{9}$$

where for the uncomplicated invariants the combinations that work for both general ODEs in equation (8) are easy to discover. For the non-normal subgroup with group generator U_3 the first-order differential invariant, y, and the path-curve, x, are $y = zu_z$ and $x = u$. Another choice of variables in terms of the invariants could give a differential equation with more arbitrary functions. For example $x = e(u)$ and $y = f(u)zu_z$ would introduce more arbitrary functions. Equivalently, one can introduce a change of variables in (10) by setting x equal to an arbitrary function of a new variable and changing y appropriately. For the compilation of an extensive set of look-up tables one might choose various combinations of y and x in terms of the invariants but that is not done here. Substituting these into (9) where we restrict our attention to the particular form with the first-order derivative to the first power, we find

$$\frac{dy}{dx} = \frac{g(x - y)}{xy} + 1 \tag{10}$$

where g is an arbitrary function of its argument. For special forms of g, equation (10) becomes Abel's equation. The other choice of differential invariant and path curve for the normal subgroup with group generator U_g is $\bar{y} = u - zu_z$ and $\bar{x} = u/z$. The differential equation (9) becomes upon substitution

$$\frac{\mathrm{d}\bar{y}}{\mathrm{d}\bar{x}} = \frac{g(\bar{y})}{\bar{x}\bar{y}} \tag{11}$$

which can be integrated by separation of variables. The variable change to convert (10) to (11) is found by solving for z as a non-local function of x and y. The expression $\bar{y} = u - zu_z$ is solved for z where u/z is replaced by $\bar{x}$. The results for x and y are

$$x = \frac{\bar{x}}{\int \mathrm{d}\bar{x}/\bar{y} + C} \qquad y = \frac{\bar{x}}{\int \mathrm{d}\bar{x}/\bar{y} + C} - \bar{y}. \tag{12}$$

The inverse coordinate transformation is

$$\bar{y} = x - y \qquad \text{and} \qquad \bar{x} = \bar{C}x \exp\left(-\int \frac{\mathrm{d}x}{y}\right) \tag{13}$$

as can be seen by comparing the arguments of the function g in (10) and (11) and by setting $x/z = \bar{x}$ and $y = \mathrm{d}x/\mathrm{d}\ln z$ which gives upon integration z as a non-local function of x and y. The solutions for x and y are then found by substituting the solution for $\bar{x}$ and changing the integration variable to $\bar{y}$ in (12) from (11). The solutions for x and y as parametric functions in $\bar{y}$ are

$$x = \frac{h_3(\bar{y})}{\int (h_3(\bar{y})\,\mathrm{d}\bar{y})/g(\bar{y}) + C} \qquad y = \frac{h_3(\bar{y})}{\int (h_3(\bar{y})\,\mathrm{d}\bar{y})/g(\bar{y}) + C} - \bar{y} \tag{14}$$

for

$$h_3(\bar{y}) = \exp\left[\int \frac{\bar{y}\,\mathrm{d}\bar{y}}{g(\bar{y})}\right].$$

For many functions $g(y)$ the integrals can be performed and the expression for x as a function of $\bar{y}$ inverted to give $\bar{y}$ as a function of x.

4. Results for the hidden symmetries

The results for the hidden symmetries of the first-order ODEs are summarized in this section. The higher-order ODE is a second-order ODE invariant under a non-Abelian two-parameter subgroup of the projective group. Given are the general forms of the first-order ODE with hidden symmetries, restricted to the first-order derivative in the first power. The variable transformation to the higher-order ODE is noted as are the variable transformations between the two first-order ODEs. The solutions for $y(x)$ in cases $\mathrm{I}_{a,b}$ and $\mathrm{II}_{a,b}$ and parametric solutions for cases $\mathrm{III}_{a,b}$ and $\mathrm{IV}_{a,b}$ are given.

The results are indicated in the tables. In table 1 for each of the eight cases noted in column 1 the two group generators for the two-parameter group are tabulated in column 2. The group generator is U_α (U_β) corresponding to the normal (non-normal) subgroup. The constant k in the commutator given by (3) is 1 for the cases $\mathrm{I}_{a,b}$ and $\mathrm{II}_{a,b}$, -1 for the cases $\mathrm{III}_{a,b}$ and $\mathrm{IV}_{a,b}$. The general form of the second-order ODE invariant under the two-parameter group of Lie point transformations is presented in

Table 1. The group generators U_α and U_β for the normal and non-normal subgroups respectively are listed in column 2 and the general forms of the second-order ODE invariant under the 2-parameter subgroup of the projective group are given in column 3. The eight 2-parameter subgroups are indicated in column 1 for this and all tables.

Case	Group generators	Second-order ODEs
I_a	$U_\alpha=\partial_z,\quad U_\beta=z\partial_z$	$F\left(u,\dfrac{u_{zz}}{u_z^2}\right)=0$
I_b	$U_\alpha=\partial_u,\quad U_\beta=u\partial_u$	$F\left(z,\dfrac{u_{zz}}{u_z}\right)=0$
II_a	$U_\alpha=u\partial_z,\quad U_\beta=z\partial_z$	$F\left(u,\dfrac{u_{zz}}{u_z^2(u-zu_z)}\right)=0$
II_b	$U_\alpha=z\partial_u,\quad U_\beta=u\partial_u$	$F\left(z,\dfrac{u_{zz}}{u-zu_z}\right)=0$
III_a	$U_\alpha=z^2\partial_z+zu\partial_u,\quad U_\beta=z\partial_z$	$F(u-zu_z,z^2uu_{zz})=0$
III_b	$U_\alpha=zu\partial_z+u^2\partial_u,\quad U_\beta=u\partial_u$	$F\left(\dfrac{u}{u_z}-z,\dfrac{zu^2u_{zz}}{u_z^3}\right)=0$
IV_a	$U_\alpha=z\partial_u,\quad U_\beta=z\partial_z$	$F(u-zu_z,z^2u_{zz})=0$
IV_b	$U_\alpha=u\partial_z,\quad U_\beta=u\partial_u$	$F\left(\dfrac{u}{u_z}-z,\dfrac{u^2u_{zz}}{u_z^3}\right)=0$

column 3. The 'solved' form of the second-order ODE can be calculated from the general form where the second-order differential invariant containing u_{zz} is set equal to a general function, g, of the other invariant. The 'solved' form here has the highest-order derivative to the first power. The statements about the reduction of the order of an ODE which is invariant under a Lie point group, are usually made about ODEs with the highest-order ODE appearing in the first power as a function of the lower-order derivatives and the two variables. Transcendental expressions such as were found for a carrier transport equation in semiconductors for the first-order derivative and the function [23] are not considered here and are usually excluded.

In table 2 the variable transformation for the non-normal subgroup is presented. The first-order differential invariants and invariants found from the extended group generators for U_β, the non-normal group generator, are tabulated in column 2. This is the variable transformation that reduces the order of the 'solved' form of the second-order ODE to the first-order ODE. First-order ODEs, which have the hidden symmetries, are tabulated in column 3. These are the equations, excluding case I, with no apparent invariance under one-parameter Lie point groups. The corresponding quantities for the normal subgroup are tabulated in table 3. The coordinate transformations from (x, y) of the non-normal subgroup variables to $(\bar{x}, \bar{y})$ of the normal subgroup variables and the inverse coordinate transformations are not tabulated here but are given in a technical report.

In table 3 the first-order ODEs are identical for the cases with the same Roman numeral and very similar otherwise. This similarity is not accidental; the eight cases can be classified into two types of the four possible normal forms of group generators of two-parameter groups [10]. In the normal forms one group generator is for translational invariance. The two types of normal forms found here are for non-Abelian

Table 2. The variable transformation or invariants for the non-normal subgroup are given in column 2. The first-order ODEs for the non-normal subgroup variables are listed in column 3.

Case	Non-normal variables	First-order ODEs
I_a	$y=zu_z,\quad x=u$	$\frac{\mathrm{d}y}{\mathrm{d}x}=yg(x)+1$
I_b	$y=\frac{u_z}{u},\quad x=z$	$\frac{\mathrm{d}y}{\mathrm{d}x}=yg(x)-y^2$
II_a	$y=zu_z,\quad x=u$	$\frac{\mathrm{d}y}{\mathrm{d}x}=y(x-y)g(x)+1$
II_b	$y=\frac{u_z}{u},\quad x=z$	$\frac{\mathrm{d}y}{\mathrm{d}x}=(1-xy)g(x)-y^2$
III_a	$y=zu_z,\quad x=u$	$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{g(x-y)}{xy}+1$
III_b	$y=\frac{u_z}{u},\quad x=z$	$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{y^3}{x}g\left(\frac{1}{y}-x\right)-y^2$
IV_a	$y=zu_z,\quad x=u$	$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{g(x-y)}{y}+1$
IV_b	$y=\frac{u_z}{u},\quad x=z$	$\frac{\mathrm{d}y}{\mathrm{d}x}=y^3g\left(\frac{1}{y}-x\right)-y^2$

Table 3. The variable transformation or invariants for the normal subgroup are given in column 2. The first-order ODEs for the normal subgroup variables are listed in column 3.

Case	Normal variables	First-order ODEs
I_a	$\bar{y}=u_z,\quad \bar{x}=u$	$\frac{\mathrm{d}\bar{y}}{\mathrm{d}\bar{x}}=\bar{y}g(\bar{x})$
I_b	$\bar{y}=u_z,\quad \bar{x}=z$	$\frac{\mathrm{d}\bar{y}}{\mathrm{d}\bar{x}}=\bar{y}g(\bar{x})$
II_a	$\bar{y}=\frac{u}{u_z}-z,\quad \bar{x}=u$	$\frac{\mathrm{d}\bar{y}}{\mathrm{d}\bar{x}}=-\bar{x}\bar{y}g(\bar{x})$
II_b	$\bar{y}=u-zu_z,\quad \bar{x}=z$	$\frac{\mathrm{d}\bar{y}}{\mathrm{d}\bar{x}}=-\bar{x}\bar{y}g(\bar{x})$
III_a	$\bar{y}=u-zu_z,\quad \bar{x}=\frac{u}{z}$	$\frac{\mathrm{d}\bar{y}}{\mathrm{d}\bar{x}}=\frac{g(\bar{y})}{\bar{x}\bar{y}}$
III_b	$\bar{y}=\frac{u}{u_z}-z,\quad \bar{x}=\frac{u}{z}$	$\frac{\mathrm{d}\bar{y}}{\mathrm{d}\bar{x}}=\frac{g(\bar{y})}{\bar{x}\bar{y}}$
IV_a	$\bar{y}=u-zu_z,\quad \bar{x}=z$	$\frac{\mathrm{d}\bar{y}}{\mathrm{d}\bar{x}}=-\frac{g(\bar{y})}{\bar{x}}$
IV_b	$\bar{y}=\frac{u}{u_z}-z,\quad \bar{x}=u$	$\frac{\mathrm{d}\bar{y}}{\mathrm{d}\bar{x}}=-\frac{g(\bar{y})}{\bar{x}}$

Table 4. The solutions given in column 2 for $y(x)$ for cases I and II and the parametric solutions $x(\bar{y})$ and $y(\bar{y})$ are given for cases III and IV.

Case	Solutions
I_a	$y=h_1(x)\left[\int \frac{\mathrm{d}x}{h_1(x)}+K\right]$
I_b	$\frac{1}{y}=\frac{1}{h_1(x)}\left[\int h_1(x)\,\mathrm{d}x+K\right]$
II_a	$\frac{1}{y}=\frac{1}{x}+\frac{h_2(x)}{x[x\int h_2(x)\,\mathrm{d}x/x^2+K]}$
II_b	$y=\frac{1}{x}+\frac{h_2(x)}{x[x\int h_2(x)\,\mathrm{d}x/x^2+K]}$
III_a	$x=\frac{h_3(\bar{y})}{\int h_3(\bar{y})\,\mathrm{d}\bar{y}/g(\bar{y})+K}$, $y=\frac{h_3(\bar{y})}{\int h_3(\bar{y})\,\mathrm{d}\bar{y}/g(\bar{y})+K}-\bar{y}$
III_b	$\frac{1}{x}=h_3(\bar{y})\left[\int \frac{\mathrm{d}\bar{y}}{g(\bar{y})h_3(\bar{y})}+K\right]$, $\frac{1}{y}=\bar{y}+\left(h_3(\bar{y})\left[\int \frac{\mathrm{d}\bar{y}}{g(\bar{y})h_3(\bar{y})}+K\right]\right)^{-1}$
IV_a	$x=h_4(\bar{y})\left[\int \frac{\bar{y}\,\mathrm{d}\bar{y}}{g(\bar{y})h_4(\bar{y})}+K\right]$, $y=-\bar{y}+h_4(\bar{y})\left[\int \frac{\bar{y}\,\mathrm{d}\bar{y}}{g(\bar{y})h_4(\bar{y})}+K\right]$
IV_b	$x=-h_4(\bar{y})\left[\int \frac{\bar{y}\,\mathrm{d}\bar{y}}{g(\bar{y})h_4(\bar{y})}+K\right]$, $\frac{1}{y}=\bar{y}-h_4(\bar{y})\left[\int \frac{\bar{y}\,\mathrm{d}\bar{y}}{g(\bar{y})h_4(\bar{y})}+K\right]$

groups where they are further distinguished by zero or non-zero values for δ. The δ is the determinant of the coordinate functions of the group generators. In table 3 $\delta=0$ for cases I and II and $\delta\neq 0$ for cases III and IV.

The solutions of the first-order ODEs in the normal subgroup variables are tabulated in table 4. For cases $\mathrm{I}_{a,b}$ and $\mathrm{II}_{a,b}$ the relation for $y(x)$ is direct since $\bar{x}=x$. For cases $\mathrm{III}_{a,b}$ and $\mathrm{IV}_{a,b}$ a parametric relation occurs. The h-functions in table 5 are:

$$h_1(x)=\exp\left[\int g(x)\,\mathrm{d}x\right] \qquad h_2(x)=\exp\left[-\int xg(x)\,\mathrm{d}x\right]$$
$$h_3(\bar{y})=\exp\left[\int \frac{\bar{y}\,\mathrm{d}\bar{y}}{g(\bar{y})}\right] \qquad h_4(\bar{y})=\exp\left[-\int \frac{\mathrm{d}\bar{y}}{g(\bar{y})}\right]. \tag{15}$$

The group generators for the first-order ODEs with the hidden symmetries in table 2 can be calculated but are not presented here. The group generators are non-local. The properties of non-local group generators are discussed by Olver [8].

5. Vlasov characteristic equation and reaction–diffusion equation

We consider two physical examples. The first example arises in the solution of the Vlasov equation for a collisionless plasma. The Lie symmetries of the Vlasov characteristic equation were investigated earlier [16] but the solutions were limited. The equation of motion of a one-dimensional charged particle in a space- and time-dependent electric field is the Vlasov characteristic equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2}=\frac{q_j}{m_j}E(x,t) \tag{16}$$

where x is the position, t is the time, q_j is the charge, m_j is the mass of the particle and $E(x, t)$ is the electric field and SI units are used. The Lie symmetries were calculated in (16). A very general case (the coordinate function of the x-derivative of the group generator does not depend linearly on x since these result in very complicated cases) leads to a reduced equation

$$\mathscr{V}'\frac{\mathrm{d}\mathscr{V}'}{\mathrm{d}x}=\mathscr{V}'-\frac{q_j}{m_j}\frac{\mathrm{d}U'_e}{\mathrm{d}x}(\bar{x}) \tag{17}$$

where the general expressions for $\mathscr{V}'(x, v, t)$ and $\bar{x}(x, t)$ are given elsewhere [16] and $U'_e(\bar{x})$ is an arbitrary function of $\bar{x}$ at this point. The first-order, nonlinear ODE, equation (17), was found to have solutions for (i) $N=0$ which gave an energy in these transformed variables and (ii) $N\neq 0$ and $U'_e(\bar{x})$ quadratic in x.

To find more solutions the variables have been changed to alter (17) close to the form of the ODEs in table 2. The procedure in this example is to match our first-order ODE to one in table 2. Define

$$w=\mathscr{V}'+N\bar{x} \qquad \bar{x}=\bar{x} \tag{18}$$

and interchange dependent and independent variables. Then

$$\frac{\mathrm{d}\bar{x}}{\mathrm{d}w}=(w-N\bar{x})\left(-\frac{q_j}{m_j}\frac{\mathrm{d}U'_e}{\mathrm{d}\bar{x}}(\bar{x})\right)^{-1}. \tag{19}$$

Next we choose a simple form of the potential

$$U'_e(\bar{x})=\frac{U_0}{x_0}\left(\frac{x_0}{\bar{x}}\right)^p. \tag{20}$$

Four values of p were found to give solutions; more values may. For $p=-1$ the potential is quadratic in $\bar{x}$ as was already known and this is case IV_a in the tables. For $p=0$ a linear ODE results which is case I_a. For $p=\frac{1}{2}$, equation (19) can be put into the Riccati form but it does not fit the Riccati equations in table 2. For $p=1$ we find the Bernoulli equation which is case I_b in table 2. The invariant for the $p=1$ case is

$$\mathscr{I}_j=\left[\frac{1}{\bar{x}}\exp\left[-\frac{m_j(\mathscr{V}'+N\bar{x})^2}{q_jU_0}\right]+\frac{m_jN}{q_jU_0}\int_L^{\mathscr{V}'+N\bar{x}}\exp\left(-\frac{m_jw^2}{2q_jU_0}\right)\mathrm{d}w\right] \tag{21}$$

where the limit L is a constant as is $\mathscr{I}_j$ and the other cases have similar invariants.

The second example comes from the reaction–diffusion equation [28–30] which has wide applications in plasmas, biology and chemistry. Consider a particular form of the reaction–diffusion equation

$$\frac{\partial n}{\partial t}=D\frac{\partial}{\partial x}\left(n^q\frac{\partial n}{\partial x}\right)+\alpha n^p \tag{22}$$

where $n(x, t)$ is the number density, x is the position, t is the time, D is the diffusion coefficient, and q, p and α are constants. Although similarity solutions are frequently postulated for (22), we assume that a travelling wave occurs with $n(x-vt)$ for v a constant. This dependence is allowed because of the invariance of the reaction–diffusion equation under translations in x and t. Equation (22) becomes a second-order ODE

$$D\frac{\mathrm{d}}{\mathrm{d}\xi}\left(n^q\frac{\mathrm{d}n}{\mathrm{d}\xi}\right)+v\frac{\mathrm{d}n}{\mathrm{d}\xi}+\alpha n^p=0 \tag{23}$$

for $\xi=x-vt$.

Equation (23) has solutions for a number of parameters. If $q=1-p$, this equation is invariant under the stretching (dilatations) group as well as translations in ξ. In addition for $q=1-p$ this equation can be reduced to a second-order linear ODE by a non-local transformation, $S=\int \mathrm{d}\xi/nq$. This is a generalization of the diffusion equation with $q=-2$ that was first discussed for a non-local transformation by Storm [31] and later by many others [32–34]. We consider $q=-1, p=2$ here. For those values (23) is invariant under the group represented by the group generators

$$U_c=\frac{\partial}{\partial\xi} \qquad U_d=\xi\frac{\partial}{\partial\xi}-\frac{n\partial}{\partial n}. \tag{24}$$

Since $[U_c, U_d]=U_c$, then the invariants associated with U_d are the non-normal subgroup variables. If we reduce (23) by those invariants,

$$x=\xi n \qquad y=\xi^2\frac{\mathrm{d}n}{\mathrm{d}\xi}. \tag{25}$$

We find

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2\mathrm{D}xy+\mathrm{D}y^2-vx^2y-\alpha x^4}{\mathrm{D}x(x+y)} \tag{26}$$

where the translation group symmetry has been lost. Equation (26) has a hidden symmetry which can be recovered by increasing the order of the ODE back to second order to find (23).

The order of (23) can be increased by applying the Riccati transformation,

$$n=\Gamma\frac{u_z}{u} \qquad \xi=z \tag{27}$$

where the choice of this group is suggested by the form of the invariance of (23) under the stretching group and the form of the non-local transformation. We find if $\mathrm{D}=v\Gamma$ that

$$\mathrm{D}u^2u_zu_{zzz}-\mathrm{D}u^2u_{zz}^2+\alpha\Gamma u_z^4=0 \tag{28}$$

where (28) is invariant under the groups represented by the group generators, U_1, U_3 and U_4. If we use the invariants of U_3, which are non-local subgroup variables, to reduce the order of equation (28), we find a second-order ODE which has a hidden symmetry. If we use the invariants of U_1, to reduce (28) to a second-order ODE, we find the linear ODE found by a non-local transformation of (23). In that case (28) has a type II hidden symmetry in which new symmetries appear when the order of the ODE is reduced. Type II symmetries have been analysed in unpublished work.

6. Discussion

Hidden symmetries of first-order ODEs have been analysed for the eight non-Abelian subgroups of the projective group. In general the ODEs are nonlinear. This is the first systematic study of type I hidden symmetries for nonlinear ODEs although Olver [8] considered case IV_a in a somewhat different format. The non-local coordinate transformations between the two sets of variables for the first-order ODEs and the parametric form of the solutions have been stressed here. We have not reported on the non-local group generator which was mentioned by Olver. Two physical examples have been presented. These are hidden symmetries of the equation of a charged particle in a time- and space-dependent electric field and of a reaction–diffusion equation.

The hidden symmetries of nonlinear PDEs, which are usually called potential and non-classical symmetries, have received more attention recently. However, the hidden symmetries of nonlinear ODEs are very important. In the two examples from physical problems the original equation was a PDE which reduced to an ODE. Also this paper attempts to correct the misimpression that the only symmetries of the ODEs are those found by the classical Lie group method for point symmetries or its extension to contact and generalized symmetries. With this viewpoint researchers may conclude that no analytical solutions exist when in fact they do. In addition, the forms of the variable transformations found here may suggest other non-local variable transformations for ODEs, which are not apparently invariant under a Lie point group.

Two aspects of the results should be noted. First, a class of first-order ODEs, not apparently invariant under a Lie point group, can be reduced to quadratures. The extension of the approach employed here, the construction of the general form of higher-order ODEs with more complicated hidden symmetries, seems quite feasible. Second, for a particular first-order ODE the tables offer an easy check of the ODE even though the tables are still limited since only the subgroups of the projective group have been presented. The variable transformations and general form of the solutions in terms of integrals are tabulated.

In table 2, we note that the first-order ODEs for cases $I_{a,b}$ and $II_{a,b}$ are similar. A linear ODE and a Riccati equation, which is also a Bernoulli equation, are the resulting equations for cases I_a and I_b respectively. Both can be reduced to quadratures by well-tried methods. However, the coordinate transformation from the original first-order ODE variables (x, y) to the new first-order ODE variables $(\bar{x}, \bar{y})$ is non-local. The non-local character of the variable transformation is fundamental and is clearly the attribute which gives rise to the hidden symmetries. For cases $I_{a,b}$ and $II_{a,b}$ the solution $y(x)$ is shown explicitly as integrals (they may be multiple) containing the arbitrary function $g(x)$. Besides the tables for Lie point symmetries of first-order ODEs the tables of differential equations of Kamke [35] have extensive lists of first-order ODEs for which solutions have been found. The ODEs in table 2 for cases I_a and I_b are there, of course, and the ODE for case II_b is there as well but the ODE for case II_a is not found. The exclusion of the ODE for case II_a is a bit odd since all the first-order ODEs labelled with subscript b can be transformed to the comparable ODE labelled with subscript a by changing y to $1/y$ to within a minus sign.

Also in table 2 we note that the ODEs for cases $III_{a,b}$ and $IV_{a,b}$ are similar. For the cases $III_{a,b}$ and $IV_{a,b}$ the solution can be expressed in terms of parametric functions $x(\bar{y})$ and $y(\bar{y})$. For some functions $g(\bar{y})$ the relation for $x(\bar{y})$ can be inverted and y expressed in terms of x. Parametric functions in themselves are quite useful especially with computer programs such as Mathematica. None of the ODEs for cases $III_{a,b}$ and $IV_{a,b}$ appear in their general form in the Kamke tables. These are probably absent since the general form of the solution is parametric. The approach here has the advantage over the use of Kamke's tables in that many variations of the same equation are replaced by a single differential equation. In addition the approach is more systematic and demonstrates the structure of the symmetries of those equations.

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